# Fourier Transform

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IUCr International School on Crystallography
(III Latinoamerican series),
Brazil, 2012.

# Fourier Transform

Definition

$$F(k) = FT[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$$

$$f(x) = FT^{-1}[F(k)] = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x} dk$$

# Fourier Transform

Definition

$$F(\overrightarrow{k}) = FT[f(\overrightarrow{x})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f(\overrightarrow{x})e^{-2\pi i\overrightarrow{k}\cdot\overrightarrow{x}}d^N x$$

$$f(\overrightarrow{x}) = FT^{-1}[F(\overrightarrow{k})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} F(\overrightarrow{k})e^{2\pi i\overrightarrow{k}\cdot\overrightarrow{x}}d^N k$$

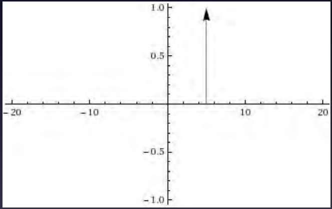
# Fourier Transform

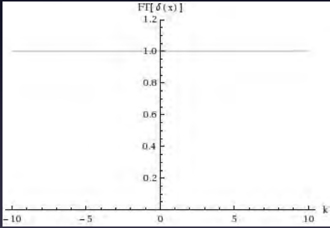
$$f(x) \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} F(k)$$

# Fourier Transform

$$f(x) = \delta(x - a)$$

$$F(k) = e^{-2\pi i k a}$$

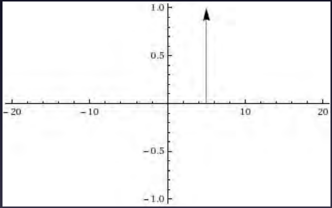


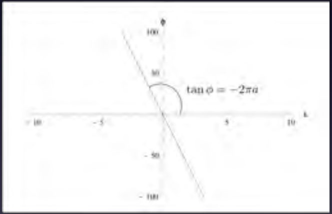



# Example 1

$$f(x) = \delta(x - a)$$

$$F(k) = e^{-2\pi i k a}$$



# The shift property

$$FT[f(x-a)] = FT[f(x)]e^{-2\pi i k a}$$

$$|FT[f(x-a)]| = |FT[f(x)]|$$

$$\phi' = \phi - 2\pi i k a$$

**Shift property**

The information regarding the shift of a function $f(x)$ is contained in the phase of the Fourier Transform

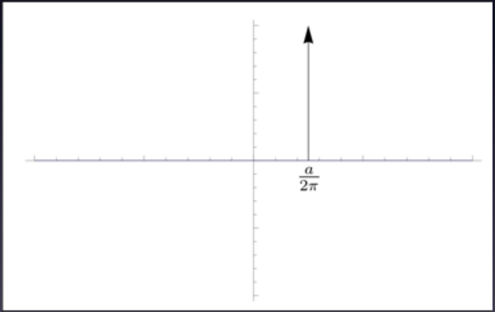
# Diffraction

**Fourier**

# Example 2

$$f(x) = e^{iax}$$

$$F(k) = \delta(k - \frac{a}{2\pi})$$

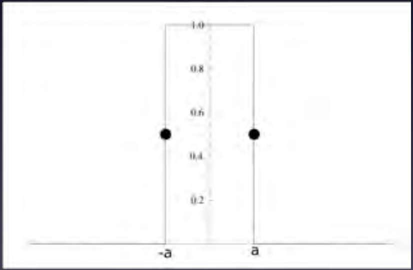



$$FT[1] = \delta(x) \quad a = 0$$

# Exercise 1

$$f(x) = rect(\frac{x}{2a})$$

$$rect(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



# The scaling property

$$FT[f(ax)] = \frac{1}{|a|} F(k/a)$$

$$|FT[f(ax)]| = \frac{1}{|a|} |F(k/a)|$$

$$FT[f(-x)] = F(-k) \quad a = -1$$

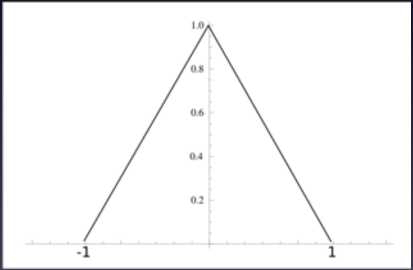
**Scaling property**

The scaling of a function $f(x)$ represents a reciprocal scaling of the Fourier Transform

# Exercise 2

$$f(x) = tri(\frac{x}{2a})$$

$$tri(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



# Parity

## Even function

$$f(x) = f(-x) \longrightarrow FT[f(-x)] = F(-k)$$

$$F(k) = F(-k)$$

## Odd function

$$f(-x) = -f(x) \longrightarrow FT[f(-x)] = F(-k)$$

$$F(-k) = -F(k)$$

## Parity property

The Fourier transform preserves parity

# Linearity property

$$h(x) = \sum_{i=1}^{N} a_i f(x)$$

$$|FT[h(x)]| = \sum_{i=1}^{N} a_i FT[f(x)]$$

# Example 3

$$f(x) = \cos ax = \frac{1}{2}\left[e^{iax} + e^{-1ax}\right] \quad w = \frac{a}{2\pi}$$

$$FT[f(x)] = \frac{1}{2}\left[FT[e^{iax} + FT[e^{-iax}]]\right]$$

$$FT[f(x)] = \frac{1}{2}\left[\delta(k - \frac{a}{2\pi}) + \delta(k + \frac{a}{2\pi})\right]$$

# Exercise 3

$$f(x) = \sin ax$$

# Periodic functions

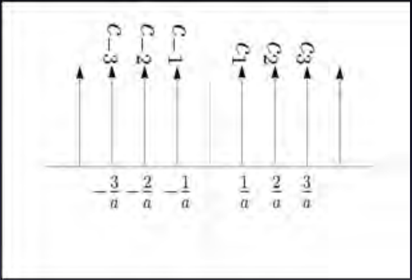
$$f(x) = f(x+T)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-2\pi i n x/T} dx$$

$$FT[f(x)] = \sum_{n=-\infty}^{\infty} c_n \delta(k - \frac{n}{T})$$

# Periodic functions




$$FT[f(x)] = \sum_{n=-\infty}^{\infty} c_n \delta(k - \frac{n}{T})$$

# Babinet Principle

$$a_1 f_1(x)+a_2 f_2(x) = Constant \longrightarrow a_1 F_1(x)+a_2 F_2(x) = Constant\delta(K)$$

$$k \neq 0 \longrightarrow a_1 F_1(x) + a_2 F_2(x) = 0$$

Babinet Principle

$$F_1(x) = -\frac{a_2}{a_1} F_2(x)$$

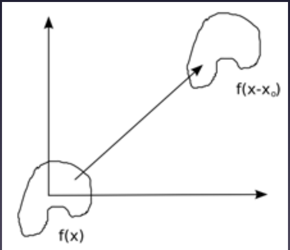
$$|F_1(x)|^2 = -|\frac{a_2}{a_1}|^2 |F_2(x)|^2$$

# Convolution theorem

## Definition

$$g(x) \otimes f(x) = \int_{\infty}^{\infty} g(y) f(x-y) dy$$

$$g(x) \otimes \delta(x-x_o) = \int_{\infty}^{\infty} g(y) \delta(y-x-x_o) dy = g(x-x_o)$$



# Convolution theorem

## Convolution theorem

$$FT[g(x) \otimes f(x)] = G(k)F(k)$$

$$FT[g(x)f(x)] = G(k) \otimes F(k)$$

# Example 4

$$h(x) = \cos axrect(x/b)$$

$$g(x) = \cos ax \;\; f(x) = rect(x/b)$$

$$G(k) = \frac{1}{2}[\delta(k - \frac{a}{2\pi}) + \delta(k + \frac{a}{2\pi})]$$

$$F(k) = b(\frac{\sin \pi kb}{\pi kb})$$

$$FT[h(x)] = b(\frac{\sin \pi kb}{\pi kb})$$

# Example 4

$$FT[h(x)] = b(\frac{\sin \pi kb}{\pi kb}) \otimes \frac{1}{2}[\delta(k - \frac{a}{2\pi}) + \delta(k + \frac{a}{2\pi})]$$

$$FT[h(x)] = \frac{b}{2}\left[\frac{\sin \pi b(k - \frac{a}{2\pi})}{\pi b(k - \frac{a}{2\pi})} + \frac{\sin \pi b(k + \frac{a}{2\pi})}{\pi b(k + \frac{a}{2\pi})}\right]$$

# Fourier transform of real functions

Theorem (Fourier transform of real functions)

*The $f(\vec{r})$ is real only and only if $F(\vec{r^*})$ is hermitian*

$$f(\vec{r}) = f^*(\vec{r}) \Longleftrightarrow F(\vec{r^*}) = F^*(-\vec{r}^*)$$

Friedel law

$$f(\vec{r}) = f^*(\vec{r}) \Rightarrow F(\vec{r^*})F^*(\vec{r^*}) = F^*(-\vec{r}^*)F(-\vec{r}^*)$$

# Friedel Law

**Friedel law**

$$f(\vec{r}) = f^*(\vec{r}) \Rightarrow F(\vec{r^*})F^*(\vec{r^*}) = F^*(-\vec{r}^*)F(-\vec{r}^*)$$