Polyhedra, Complexes, Nets, and Symmetry

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Polyhedra

- Ancient history (Greeks), closely tied to symmetry.

- With the passage of time, many changes in point of view about polyhedral structures and their symmetry. Many different definitions!!

Who knows what a polyhedron is?
**Platonic solids** (solids, convexity)

**Definition:** Set $K$ is called *convex* if every line segment joining two points of $K$ lies entirely in $K$.

**Plane:** convex polygons

**Space:** convex polyhedra — *convex hull* of finitely many points in 3-space.

(Convex hull: smallest convex set that contains the given points.)
More convex polyhedra: Archimedean solids, prisms and antiprisms
• All have regular convex polygons as faces, and all vertices are surrounded alike.
Kepler-Poinsot (star) polyhedra

What’s new here? Faces and vertex-figures can be regular star-polygons (for example, pentagrams).

Are you still sure you know what a polyhedron is? 😊
Petrie-Coxeter polyhedra (sponges)

What’s new here? Infinite polyhedra (apeirohedra)! Faces still are convex polygons! Vertex-figures allowed to be skew (non-planar) polygons! Periodic!

(Vertex-figure at vertex $x$: joins the vertices adjacent to $x$ in the order in which they occur around $x$)
Skeletal Approach

- Initiated by Branko Grünbaum (1970's), probably inspired by crystal chemistry.

- Rid the theory of the psychologically motivated block that membranes must be spanning the faces! Allow skew faces (and skew vertex-figures)! Restore the symmetry in the definition of “polyhedron”! Graph-theoretical approach!

- Basic question: what are the regular polyhedra in ordinary space? Answer: Grünbaum-Dress Polyhedra.

- Later: group theory forces skew faces and vertex-figures! General reflection groups.
• No membranes spanned into faces! Skeleton only!

• Polygon faces (simple polygons)
Who said that faces have to be finite?

zigzag polygon (planar)

helical polygon
Polyhedron (heading towards the definition)

A polyhedron \( P \) in \( \mathbb{E}^3 \) is a (finite or infinite) family of simple polygons, called faces, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit (each vertex-figure is a simple closed polygon),
- \( P \) is connected,
- each compact set (each large cube, say) meets only finitely many faces (discreteness).

All traditional polyhedra are polyhedra in this generalized sense.
What is a polygon (for our purposes)?

Finite (simple) polygon (\textit{p-gon}, with \( p \geq 3 \))

- finite sequence of \( p \) distinct points \( v_1, v_2, \ldots, v_p \), called the \textit{vertices};
- line segments \([v_1, v_2], [v_2, v_3], \ldots, [v_{p-1}, v_p] \) and \([v_p, v_1] \), called the \textit{edges}.

A finite polygon is topologically a 1-sphere!

Infinite (simple) polygon (\textit{apeirogon})

- infinite sequence of \\textit{distinct} points \( \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots \), called the \textit{vertices};
- the line segments \([v_i, v_{i+1}] \) for each integer \( i \);
- each large cube meets only finitely many of these line segments.

An infinite polygon is topologically a real line!
Some skew (non-planar) hexagons!

Four skew hexagons in the edge graph of the cube (red, blue, green, black).
Some zigzag apeirogons!

Four zigzag apeirogons in the edge graph of the square tessellation.
Some definitions

$\mathcal{F}$ any geometric figure in 3-space (possibly planar or linear)

- A symmetry of $\mathcal{F}$ is an isometry of the ambient space that leaves $\mathcal{F}$ invariant.

- The set of symmetries of $\mathcal{F}$ is a group called the symmetry group of $\mathcal{F}$ and denoted $G(\mathcal{F})$.

Isometries in the plane: reflection in a line, rotation about a point, translation, glide reflection

Isometries in 3-space: reflection in a plane, rotation about a line, rotatory reflection, screw motion (rotation followed by a translation along the axis).
Regular polygons

- Polygon $P$ is \textit{regular} if its symmetry group $G(P)$ acts transitively on the set of flags (incident vertex-edge pairs).

Then $G(P)$ is vertex-transitive (isogonal) and edge-transitive (isotoxal). But a priori flag-transitivity is stronger!

edge-transitive octagon, not regular

two vertex-transitive apeirogons, not regular
What kind of regular polygons exist?

Regular polygons in $\mathbb{E}^3$ are either

- finite planar (convex or star-) polygons,
- finite non-planar (skew) polygons,
- infinite planar zigzags,
- infinite helical polygons.

How about the symmetry groups? Generators? Later!!
POLYGONS

Figure 1

Figure 3
Helical polygons.

(a) Convex polygons

(b) Star polygons

(c) Apoline

(d) Zigzag

(a) Assenomorphic polygons

(b) Prismatic polygons
**Polyhedron** (formal definition)

A polyhedron $P$ in $\mathbb{E}^3$ is a (finite or infinite) family of (finite or infinite) simple polygons, called *faces*, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit (each vertex-figure* is a simple closed polygon),
- $P$ is connected (the edge graph is connected),
- each compact set (each large cube, say) meets only finitely many faces (discreteness)†.

*Vertex-figure at vertex $x$ of $P$: polygon whose vertices are the vertices of $P$ adjacent to $x$, and whose edges join two vertices $u$ and $v$ if and only if $[x,u]$ and $[x,v]$ are edges of a common face of $P$ at $x$.

†Consequence: Vertex-figures are finite!
Vertex-figure of the Petrie-Coxeter polyhedron \{4, 6\mid 4\}

Vertex-figures are skew hexagons! Faces are squares (square cycles)!
All are regular!
Highly symmetric polyhedra $P$

- Faces are finite (planar or skew) or infinite (helical or zig-zags) polygons! Vertex-figures are finite (planar or skew) polygons!

- Polyhedron $P$ is regular if its symmetry group $G(P)$ is transitive on the set of flags.

  *Flag*: incident vertex-edge-face triple

Explicitly: any incident vertex-edge-face triple can be mapped to any incident vertex-edge-face triple by a symmetry of $P$. 
Exercise 1: If a polyhedron $P$ is regular then $G(P)$ is also transitive, separately, on the
- vertices (isogonal), edges (isotoxal), faces (isohedral),
- vertex-edge pairs, vertex-face-pairs, and edge-face pairs!

But a priori flag-transitivity is stronger! (Tilings by rhombi can be transitive on vertices, edges, faces, but not regular!)

Consequence: congruent faces, congruent edges, congruent vertex-figures!!
Regular polyhedra

Must be able to map each flag to each of its adjacent flags!

Two flags are called *adjacent* if they differ in exactly one element.
**Exercise 2:** If $P$ is any polyhedron, then $P$ is *strongly flag-connected*, meaning that for any two flags $\Phi$ and $\Psi$ of $P$ there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_n = \Psi$, all containing the elements common to both $\Phi$ and $\Psi$ (if any), such that $\Phi_j$ and $\Phi_{j+1}$ are adjacent for all $j = 0, \ldots, n-1$.

**Exercise 3:** If $P$ is any polyhedron and $\Phi$ and $\Psi$ are flags of $P$, then there exists at most one symmetry of $P$ mapping $\Phi$ to $\Psi$. In particular, if $P$ is regular, there exists exactly one symmetry of $P$ mapping $\Phi$ to $\Psi$. 
Hierarchy of Symmetry

- Top of hierarchy: regular polyhedra!! They have maximum possible symmetry, by Exercise 3(b).

- P is chiral if \( G(P) \) has two orbits on the flags such that adjacent flags are in distinct orbits.

No longer able to map a flag to its adjacent flags!
• $P$ is Archimedean (or uniform) if $G(P)$ is vertex-transitive and $P$ has regular polygons as faces.

**Analogues of the Archimedean solids!!** Vertex-figures congruent! Faces all regular but possibly of several kinds.

• Other interesting classes! Isohedral, isotoxal, isohedral polyhedra, for example!
Some examples of regular polyhedra

The Petrie dual of the cube. A regular polyhedron with 8 vertices, 12 edges, 4 skew hexagonal faces. Type \{6, 3\}. 
Petrie duals (Petrials) of regular polyhedra $P$

**Petrie polygon of $P$:** path along the edges of $P$ such that any two, but no three, consecutive edges lie in a common face of $P$.

**Petrie dual $P^\pi$ of $P$:** same vertices and same edges as $P$, but the faces are the Petrie polygons of $P$. (Almost always again a polyhedron.)

- $(P^\pi)^\pi = P$, that is, the Petrie dual of the Petrie dual is the original polyhedron.

- $G(P^\pi) = G(P)$, but generators change as follows:
  
  $$(R_0, R_1, R_2) \rightarrow (R_0R_2, R_1, R_2)$$
The Petrie dual of the square tessellation. An infinite regular polyhedron with zig-zag faces. Type \(\{\infty, 4\}\).
The helix-faced regular polyhedron $\{\infty, 3\}^{(b)}$. 
The helix-faced polyhedron \(\{\infty, 3\}^{(b)}\)
Symmetry group of a regular polyhedron

- Generated by reflections $R_0, R_1, R_2$ in points, lines, or planes.

Note: Every isometry of $\mathbb{E}^3$ which is an involution (its order is 2) is a reflection in a point, line, or plane!

(A reflection in a line is a halfturn about the line!)
Exercise 4: Let $P$ be a regular polyhedron, and let $\Phi$ be a fixed flag (base flag) of $P$. Let $R_0$, $R_1$, and $R_2$, resp., denote the symmetry of $P$ that maps $\Phi$ to its adjacent flag differing from $\Phi$ in the vertex, edge, or face. Then $R_0, R_1, R_2$ are involutory generators of $G(P)$.

Note

- $\langle R_0, R_1 \rangle$ is the symmetry group of the base facet.

- $\langle R_1, R_2 \rangle$ is the symmetry group of the vertex-figure at the base vertex.
Symmetry groups of regular polygon $P$

$G(P) = \langle R_0, R_1 \rangle$, where $R_0, R_1$ are reflections in points, lines or planes.

Mirror vector $(m_0, m_1)$, where $m_i$ is the dimension of the mirror of $R_i$)

Regular polygons in $\mathbb{E}^3$ with mirror vector:

- finite planar (convex or star-) polygons, $(m_0, m_1) = (1, 1)$ (in the plane),
- finite non-planar (skew) polygons, $(m_0, m_1) = (2, 1)$ (in space),
- infinite planar zigzags, $(m_0, m_1) = (1, 0)$ (in the plane),
- infinite helical polygons, $(m_0, m_1) = (1, 1)$ (in space).
How to recover polyhedron $P$ from $G(P) = \langle R_0, R_1, R_2 \rangle$

$R_0$, $R_1$ and $R_2$ are the reflections in the planes bounding the tetrahedron shown, opposite to vertex 0, 1 and 2, resp.
How to recover polyhedron $P$ from $G(P)$

Wythoff’s construction! Builds an orbit structure!

– First construct a flag $\Phi = \{v, e, f\}$ of $P$, the base flag!

– Pick an initial vertex $v$ as base vertex.

– Base edge $e := [v, R_0(v)]$.

– Base face $f := \{g(e) \mid g \in \langle R_0, R_1 \rangle\}$.

– The vertex-set, edge-set and face-set of $P$ are the orbits of $v$, $e$ and $f$, respectively.

Needed: classification of suitable triples of reflections $R_0, R_1, R_2$ in 3-space!
Regular Polyhedra in $\mathbb{R}^3$
(Grünbaum, 1970’s, Dress, 1981; McMullen & S., 1997)

18 finite polyhedra: 5 Platonic, 4 Kepler-Poinsot, 9 Petrials.
(2 full tetrahedral symmetry, 4 full octahedral, 12 full icosahedral)
Icosahedron: \( \{3, 5\}^\pi \) is of type \( \{10, 5\} \)
Finite regular polyhedra

18 finite (5 Platonic, 4 Kepler-Poinsot, 9 Petrials)

tetrahedral \{3, 3\} \xrightarrow{\pi} \{4, 3\}_3

octahedral \{6, 4\}_3 \xrightarrow{\pi} \{3, 4\} \xrightarrow{\delta} \{4, 3\} \xrightarrow{\pi} \{6, 3\}_4

icosahedral \{10, 5\} \xrightarrow{\pi} \{3, 5\} \xrightarrow{\delta} \{5, 3\} \xrightarrow{\pi} \{10, 3\}
\downarrow \varphi_2 \quad \uparrow \varphi_2

\{6, \frac{5}{2}\} \xrightarrow{\pi} \{5, \frac{5}{2}\} \xrightarrow{\delta} \{\frac{5}{2}, 5\} \xrightarrow{\pi} \{6, 5\}
\downarrow \varphi_2 \quad \uparrow \varphi_2

\{\frac{10}{3}, 3\} \xrightarrow{\pi} \{\frac{5}{2}, 3\} \xrightarrow{\delta} \{3, \frac{5}{2}\} \xrightarrow{\pi} \{\frac{10}{3}, \frac{5}{2}\}

duality \delta: R_2, R_1, R_0; \quad \text{Petrie } \pi: R_0R_2, R_1, R_0; \quad \text{facetting } \varphi_2: R_0, R_1R_2R_1, R_2
Facetting operation $\varphi_2$ on $P$

**New polyhedron** $P^{\varphi_2}$: same vertices and edges as $P$, but the new faces of $P$ are the **2-holes** of $P$.

**2-hole of** $P$: path along edges which leaves a vertex by the *second* edge from which it entered, always in the same sense (on the left, say).

Example: $P = \{3, 5\}$
Icosahedron: \( \{3, 5\} \varphi^2 = \{5, \frac{5}{2}\} \), a KP polyhedron
30 apeirohedra (infinite polyhedra)! Crystallographic groups!
6 planar (3 regular tessellations and their Petrials)

The Petrie dual $\{4,4\}^\pi$, of type $\{\infty,4\}$.
12 reducible apeirohedra.

Blends of a planar polyhedron and a line segment

Square tessellation blended with the line segment. Symbol \{4, 4\}#{ }. 
12 reducible apeirohedra.

Same blend, different ratio between components

Square tessellation blended with the line segment. Symbol \{4, 4\}#\{\}. 
Blends of a planar polyhedron and a line tessellation

The square tessellation blended with a line tessellation. Symbol $\{4,4\} \neq \{\infty\}$. Each vertical column over a square is occupied by one helical facet spiraling around the column.
Same blend, different ratio

The square tessellation blended with a line tessellation. Symbol \( \{4, 4\} \# \{\infty\} \).
Each vertical column over a square is occupied by one helical facet spiraling around the column.
12 irreducible apeirohedra.

\[
\{\infty, 4\}_{6,4} \xrightarrow{\pi} \{6, 4|4\} \xrightarrow{\delta} \{4, 6|4\} \xrightarrow{\pi} \{\infty, 6\}_{4,4}
\]

\[
\sigma \downarrow \quad \downarrow \eta
\]

\[
\{\infty, 4\}_{*3} \xrightarrow{\varphi_2} \{\infty, 3\}^{(a)}
\]

\[
\pi \uparrow \quad \uparrow \pi
\]

\[
\{6, 4\}_6 \xrightarrow{\delta} \{4, 6\}_6 \xrightarrow{\varphi_2} \{\infty, 3\}^{(b)}
\]

\[
\sigma\delta \downarrow \quad \downarrow \eta
\]

\[
\{\infty, 6\}_{6,3} \xrightarrow{\pi} \{6, 6|3\}
\]

\eta: R_0R_1R_0, R_2, R_1; \quad \sigma = \pi\delta\eta\pi\delta: R_1, R_0R_2, (R_1R_2)^2; \quad \varphi_2: R_0, R_1R_2R_1, R_2
Petrie-Coxeter polyhedra (sponges)
The Petrie dual of the Petrie-Coxeter polyhedron \{6,4\mid4\}. Alternative notation \{\infty,4\}_{6,4}.

Not every regular polyhedron has a (geometric) dual! For example, \{\infty,4\}_{6,4} does not!
The polyhedron $\{6, 6\}_4$ derived from $\{4, 6|4\}$ via $\eta$

- Bicolor the vertices of $\{4, 6|4\}$.
- Vertex-figures at vertices in one class give faces of $\{6, 6\}_4$.
- New polyhedron $\{6, 6\}_4$ has planar vertex-figures.
The helix-faced regular polyhedron \( \{ \infty, 3 \}^{(b)} \). Its Petrie dual is \( \{ \infty, 3 \}^{(a)} \). Neither has a (geometric) dual!

Symmetry group of \( \{ \infty, 3 \}^{(b)} \) requires the single extra relation

\[
(R_0 R_1)^4 (R_0 R_1 R_2)^3 = (R_0 R_1 R_2)^3 (R_0 R_1)^4.
\]
Breakdown by mirror vector (for reflection generators $R_0, R_1, R_2$)

(vector $(m_0, m_1, m_2)$, where $m_i$ is the dimension of the mirror of $R_i$)

<table>
<thead>
<tr>
<th>mirror vector</th>
<th>{3, 3}</th>
<th>{3, 4}</th>
<th>{4, 3}</th>
<th>faces</th>
<th>vertex-figures</th>
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<tr>
<td>(2,1,2)</td>
<td>{6, 6</td>
<td>3}</td>
<td>{6, 4</td>
<td>4}</td>
<td>{4, 6</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>{∞, 6}4,4</td>
<td>{∞, 4}6,4</td>
<td>{∞, 6}6,3</td>
<td>helical</td>
<td>skew</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>{6, 6}4</td>
<td>{6, 4}6</td>
<td>{4, 6}6</td>
<td>skew</td>
<td>planar</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>{∞, 3}$_{(a)}$</td>
<td>{∞, 4}..,*3</td>
<td>{∞, 3}$_{(b)}$</td>
<td>helical</td>
<td>planar</td>
</tr>
</tbody>
</table>

Last row: polyhedra occur in two enantiomorphic forms, yet geometrically regular!

Presentations for the symmetry group are known. The fine Schläfli symbol signifies defining relations. Extra relations specify order of $R_0R_1R_2$, $R_0R_1R_2R_1$, or $R_0(R_1R_2)^2$. 
Chirality – Origin of the Term

First used by Lord Kelvin in 1893:

*I call any geometrical figure, or group of points, “chiral”, and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself.*

“chiral” comes from the greek word χειρ (kheir), which means “hand”.

![Hand and Shell Diagram](image)
Goal

Study chirality in the presence of very high symmetry!

Snub cube

Exhibits chirality, but for our purposes not symmetric enough!!
Chiral Polyhedra in $\mathbb{E}^3$

- Two flag orbits, with adjacent flags in different orbits.
- Local picture

$T = S_1 S_2$

- $P$ must have symmetries $S_1$ and $S_2$ such that $S_1$ cyclically permutes the vertices in one face, and $S_2$ cyclically permutes the faces that meet at a vertex of that face.
• Then $G(P) = \langle S_1, S_2 \rangle$, and $S_1^p = S_2^q = (S_1 S_2)^2 = 1$ if $P$ is of type \{p, q\}.

![Diagram](image)

Note: $S_1, S_2$ are NOT geometric rotations in general, although they act like rotations would!

• Maximal “rotational” symmetry but no “reflexive” symmetry! No $R_i$’s! Irreflexible!

(Regularity: maximal “reflexive” symmetry.)
NOTES

• No classical examples! Convex polyhedra cannot be chiral!

Convex polyhedra: if a polyhedron looks chiral then it is actually regular!! If symmetries like $S_1$ and $S_2$ are present, then reflections like $R_0, R_1, R_2$ are also present.

• Can show: No finite chiral polyhedra in $\mathbb{E}^3$!

Depressing!!

• No planar or blended chiral polyhedra in $\mathbb{E}^3$!

Depressing!!
GOOD NEWS: chiral polyhedra do exist!

- Classification breaks down into
  - polyhedra with finite faces and
  - polyhedra with infinite faces!

- Three very large 2-parameter families of chiral polyhedra of each kind!

- All polyhedra are infinite! Faces must be skew regular polygons (if finite) or helical regular polygons (if infinite).
Chiral polyhedron $P(1, 0)$, type $\{6, 6\}$. Neighborhood of a single vertex.
Chiral polyhedron $Q(1, 1)$, type $\{4, 6\}$. Neighborhood of a single vertex.
Three Classes of Finite-Faced Chiral Polyhedra
($S_1, S_2$ rotatory reflections, hence skew faces and skew vertex-figures.)

<table>
<thead>
<tr>
<th>Schlafli</th>
<th>{6, 6}</th>
<th>{4, 6}</th>
<th>{6, 4}</th>
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<tr>
<td>Notation</td>
<td>$P(a, b)$</td>
<td>$Q(c, d)$</td>
<td>$Q(c, d)^*$</td>
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<tr>
<td>Param.</td>
<td>$a, b \in \mathbb{Z},$</td>
<td>$c, d \in \mathbb{Z},$</td>
<td>$c, d \in \mathbb{Z},$</td>
</tr>
<tr>
<td></td>
<td>$(a, b) = 1$</td>
<td>$(c, d) = 1$</td>
<td>$(c, d) = 1$</td>
</tr>
<tr>
<td></td>
<td>geom. self-dual</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$P(a, b)^* \cong P(a, b)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular cases</td>
<td>$P(a, -a) = {6, 6}_4$</td>
<td>$Q(c, 0) = {4, 6}_6$</td>
<td>$Q(c, 0)^* = {6, 4}_6$</td>
</tr>
<tr>
<td></td>
<td>$P(a, a) = {6, 6</td>
<td>3}$</td>
<td>$Q(0, d) = {4, 6</td>
</tr>
</tbody>
</table>

Each extended family contains two regular polyhedra (for these parameter values the faces or vertex-figures become flat).
Chiral polyhedron $Q(1, 1)$. Skew squares, six at each vertex. Vertex set is $\Lambda_3$. (All models built and photographed by Daniel Pellicer.)
Another view of the chiral polyhedron $Q(1, 1)$. 
Still another view of $Q(1, 1)$. 
Yet another view of $Q(1,1)$. 
Even more on $Q(1,1)$ .... :)  

Skeletal Geometry!
Three Classes of Helix-Faced Chiral Polyhedra

($S_1$ screw motion, $S_2$ rotation; helical faces and planar vertex-figures.)

<table>
<thead>
<tr>
<th>Schlӓfli</th>
<th>${\infty, 3}$</th>
<th>${\infty, 3}$</th>
<th>${\infty, 4}$</th>
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<tr>
<td>Notat.</td>
<td>$P_1(a, b)$</td>
<td>$P_2(c, d)$</td>
<td>$P_3(c, d)^*$</td>
</tr>
<tr>
<td>Param.</td>
<td>$a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$</td>
<td>$c, d \in \mathbb{R}$, $(c, d) \neq (0, 0)$</td>
<td>$c, d \in \mathbb{R}$, $(c, d) \neq (0, 0)$</td>
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<td>Helices</td>
<td>triangles</td>
<td>squares</td>
<td>triangles</td>
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<tr>
<td>over</td>
<td>$P(a, b)^{\varphi 2}$</td>
<td>$Q(c, d)^{\varphi 2}$</td>
<td>$Q^*(c, d)^{\kappa}$</td>
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<tr>
<td>Regular</td>
<td>$P_1(a, -a) = {\infty, 3}^{(a)}$</td>
<td>$P_2(c, 0) = {\infty, 3}^{(b)}$</td>
<td>$P_3(0, d) = {\infty, 4}..,*_3$ (self-Petrie)</td>
</tr>
<tr>
<td></td>
<td>$P_1(a, a) = {3, 3}$</td>
<td>$P_2(0, d) = {4, 3}$</td>
<td>$P_3(c, 0) = {3, 4}$</td>
</tr>
</tbody>
</table>

Each extended family contains two regular polyhedra, one finite and one infinite. Helices collapse or vertex-stars become planar.
Chiral polyhedron $P_1(0,1)$. Helical faces over triangles, three at each vertex. Photo taken in the direction of a helix; triangular projection of a helical face visible.
Four views of the chiral polyhedron $P_1(0, 1)$. 
Chiral polyhedron $P_2(1,1)$. Helical faces over squares, three at each vertex. Photo taken in the direction of a helix.
Four more views of the chiral polyhedron $P_2(1,1)$. 
Remarkable facts about chiral polyhedra

• Essentially: any two finite-faced polyhedra of the same type are combinatorially non-isomorphic.
  
  \[ P(a, b) \cong P(a', b') \text{ iff } (a', b') = \pm(a, b), \pm(b, a). \]

  \[ Q(c, d) \cong Q(c', d') \text{ iff } (c', d') = \pm(c, d), \pm(-c, d). \]

• Finite-faced polyhedra are combinatorially chiral! [Pellicer & Weiss 2009]

• Helix-faced polyhedra combinatorially regular! Combinatorially only 3 polyhedra! Chiral helix-faced polyhedra are deformations of regular helix-faced polyhedra! [P&W 2009]

• Chiral helix-faced polyhedra unravel Platonic solids! Coverings

  \[ \{\infty, 3\} \mapsto \{3, 3\}, \quad \{\infty, 3\} \mapsto \{4, 3\}, \quad \{\infty, 4\} \mapsto \{3, 4\}. \]
More polygons on an edge ....

Neighborhood of a vertex in $K_4(1,2)$: 6 faces at an edge; 12 at a vertex (octah. vertex-fig.). All Petrie polygons of every other cube. Net **pcu**.
A polygonal complex $K$ in $\mathbb{E}^3$ is a family of simple polygons, called *faces*, such that

- each edge of a face is an edge of exactly $r$ faces ($r \geq 2$);
- the vertex-figure at each vertex is a connected graph, possibly with multiple edges;
- the edge graph of $K$ is connected;
- each compact set meets only finitely many faces (discreteness).

$K$ is *regular* if its geometric symmetry group $G(K)$ is transitive on the flags of $K$.

(flag: incident vertex-edge-face triple)
Examples

- Regular polyhedra \((r = 2)\). There are 48.
- All squares of the cubical tessellation \((r = 4)\).

Vertex-figure: octahedral graph!
$\kappa_1(1, 2)$: four tetragons on an edge. Petrie polygons of tetrahedra inscribed in cubes, in an alternating fashion. The net is fcu.
$\mathcal{K}_5(1, 2)$: 4 faces at an edge, 8 at a vertex (double square as vertex-figure). One Petrie-polygon for each cube. Net is \textit{nbo} (Niobium Monoxide, NbO).
Case: group $G(K)$ not simply flag-transitive

- $K$ is the 2-skeleton of a certain rank 4 structure in $\mathbb{E}^3$, called a regular 4-apeirotope. There are eight such rank 4 structures, contributing four regular polygonal complexes!

Eight regular 4-apeirotopes in $\mathbb{E}^3$ (in pairs of Petrie duals). Two have square faces. The others have zigzag faces. Complexes have face mirrors!

\[
\begin{align*}
\{4, 3, 4\} &\quad \{4, 6 \mid 4\}, \{6, 4\}_3 \\
\text{apeir}\{3, 3\} = \{\{\infty, 3\}_6 \# \{\}, \{3, 3\}\} &\quad \{\infty, 4\}_4 \# \{\infty\}, \{4, 3\}_3 = \text{apeir}\{4, 3\}_3 \\
\text{apeir}\{3, 4\} = \{\{\infty, 3\}_6 \# \{\}, \{3, 4\}\} &\quad \{\infty, 6\}_3 \# \{\infty\}, \{6, 4\}_3 = \text{apeir}\{6, 4\}_3 \\
\text{apeir}\{4, 3\} = \{\{\infty, 4\}_4 \# \{\}, \{4, 3\}\} &\quad \{\infty, 6\}_3 \# \{\infty\}, \{6, 3\}_4 = \text{apeir}\{6, 3\}_4
\end{align*}
\]

(P.McMullen & E.S., Abstract Regular Polytopes, CUP, 2002)
Case: group $G(K)$ simply flag-transitive

• Finite complexes must be polyhedra (18 examples).

• $G(K)$ affinely reducible: only polyhedra ($6 + 12$).

• $G(K)$ an affinely irreducible crystallographic group!
  Further, $G(K) = \langle G_0, G_1, G_2 \rangle$, with

<table>
<thead>
<tr>
<th>$G_0$</th>
<th>$G_1$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle R_0 \rangle$</td>
<td>$\langle R_1 \rangle$</td>
<td>$C_r, D_{r/2}$</td>
</tr>
<tr>
<td>$R_0$ reflection in point, line or plane</td>
<td>$R_1$ reflection in line or plane</td>
<td>cyclic or dihedral, of order $r$</td>
</tr>
</tbody>
</table>
The 21 simply flag-transitive regular polygonal complexes in $\mathbb{E}^3$ which are not polyhedra, and their nets.

<table>
<thead>
<tr>
<th>Complex</th>
<th>$G_2$</th>
<th>$r$</th>
<th>Face</th>
<th>Vertex-Figure</th>
<th>Vertex Set</th>
<th>$G^*$</th>
<th>Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1(1,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$4_s$</td>
<td>cuboctahedron</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K_2(1,2)$</td>
<td>$C_3$</td>
<td>3</td>
<td>$4_s$</td>
<td>cube</td>
<td>$\Lambda_3$</td>
<td>[3, 4]</td>
<td>bcu</td>
</tr>
<tr>
<td>$K_3(1,2)$</td>
<td>$D_3$</td>
<td>6</td>
<td>$4_s$</td>
<td>double cube</td>
<td>$\Lambda_3$</td>
<td>[3, 4]</td>
<td>bcu</td>
</tr>
<tr>
<td>$K_4(1,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$6_s$</td>
<td>octahedron</td>
<td>$\Lambda_1$</td>
<td>[3, 4]</td>
<td>pcu</td>
</tr>
<tr>
<td>$K_5(1,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$6_s$</td>
<td>double square</td>
<td>$V$</td>
<td>[3, 4]</td>
<td>nbo</td>
</tr>
<tr>
<td>$K_6(1,2)$</td>
<td>$D_4$</td>
<td>8</td>
<td>$6_s$</td>
<td>double octah.</td>
<td>$\Lambda_1$</td>
<td>[3, 4]</td>
<td>pcu</td>
</tr>
<tr>
<td>$K_7(1,2)$</td>
<td>$D_3$</td>
<td>6</td>
<td>$6_s$</td>
<td>double tetrah.</td>
<td>$W$</td>
<td>[3, 4]</td>
<td>dia</td>
</tr>
<tr>
<td>$K_8(1,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$6_s$</td>
<td>cuboctahedron</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K_1(1,1)$</td>
<td>$D_3$</td>
<td>6</td>
<td>$\infty_3$</td>
<td>double cube</td>
<td>$\Lambda_3$</td>
<td>[3, 4]</td>
<td>bcu</td>
</tr>
<tr>
<td>$K_2(1,1)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$\infty_3$</td>
<td>double square</td>
<td>$V$</td>
<td>[3, 4]</td>
<td>nbo</td>
</tr>
</tbody>
</table>

nbo = net of Niobium Monoxide, NbO
The 21 complexes and their nets (cont.).

<table>
<thead>
<tr>
<th>Complex</th>
<th>$G_2$</th>
<th>$r$</th>
<th>Face</th>
<th>Vertex-Figure</th>
<th>Vertex Set</th>
<th>$G^*$</th>
<th>Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_3(1,1)$</td>
<td>$D_4$</td>
<td>8</td>
<td>$\infty_3$</td>
<td>double octah.</td>
<td>$\Lambda_1$</td>
<td>[3, 4]</td>
<td>pcu</td>
</tr>
<tr>
<td>$K_4(1,1)$</td>
<td>$D_3$</td>
<td>6</td>
<td>$\infty_4$</td>
<td>double tetrah.</td>
<td>$W$</td>
<td>[3, 4]</td>
<td>dia</td>
</tr>
<tr>
<td>$K_5(1,1)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$\infty_4$</td>
<td>ns-cuboctah.</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K_6(1,1)$</td>
<td>$C_3$</td>
<td>3</td>
<td>$\infty_4$</td>
<td>tetrahedron</td>
<td>$W$</td>
<td>[3, 4]$^+$</td>
<td>dia</td>
</tr>
<tr>
<td>$K_7(1,1)$</td>
<td>$C_4$</td>
<td>4</td>
<td>$\infty_3$</td>
<td>octahedron</td>
<td>$\Lambda_1$</td>
<td>[3, 4]$^+$</td>
<td>pcu</td>
</tr>
<tr>
<td>$K_8(1,1)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$\infty_3$</td>
<td>ns-cuboctah.</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K_9(1,1)$</td>
<td>$C_3$</td>
<td>3</td>
<td>$\infty_3$</td>
<td>cube</td>
<td>$\Lambda_3$</td>
<td>[3, 4]$^+$</td>
<td>bcu</td>
</tr>
<tr>
<td>$K(0,1)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$\infty_2$</td>
<td>ns-cuboctah.</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K(0,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$\infty_2$</td>
<td>cuboctah.</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K(2,1)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$6_c$</td>
<td>ns-cuboctah.</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
<tr>
<td>$K(2,2)$</td>
<td>$D_2$</td>
<td>4</td>
<td>$3_c$</td>
<td>cuboctahedron</td>
<td>$\Lambda_2$</td>
<td>[3, 4]</td>
<td>fcu</td>
</tr>
</tbody>
</table>

$$V := \mathbb{Z}^3 \setminus ((0, 0, 1) + \Lambda_{(1,1,1)}), \quad W := 2\Lambda_{(1,1,0)} \cup ((1, -1, 1) + 2\Lambda_{(1,1,0)})$$
Edge-graph (net) of $K_7(1,2)$: diamond net, modeling the diamond crystal. (Carbon atoms sit at the vertices, and bonds between adjacent atoms are represented by edges. The “hexagonal rings” are the faces of $K_7(1,2)$.)

Edges of $K_7(1,2)$ run along main diagonals of cubes in $\{4,3,4\}$. Six skew hexagonal faces around an edge ($r = 6$). Vertex-figure is the double edge-graph of the tetrahedron (so 12 faces meet at a vertex).
**Nets**

*Net* $\mathcal{N}$: geometric graph in 3-space which is 3-periodic (translation symmetries in 3 independent directions)

*Uninodal net* $\mathcal{N}$: symmetry group $G(\mathcal{N})$ of $\mathcal{N}$ acts transitively on the vertices (nodes) of $\mathcal{N}$

*Coordination figure of* $\mathcal{N}$ *at vertex* $v$: convex hull of the neighbors of $v$ in $\mathcal{N}$

Examples of uninodal nets: edge graphs of most regular polygonal complexes and chiral polyhedra are uninodal nets. (Convex hull of the vertex-figure at $v$ is the coordination figure at $v$.)
**Regular net $\mathcal{N}$**

- $\mathcal{N}$ is uninodal,
- coordination figures are convex regular polygons or Platonic solids,
- each vertex stabilizer (site symmetry group) in $G(\mathcal{N})$ contains the rotation symmetry group of the coordination figure.

(Rotation symmetry group of a regular convex polygon is taken relative to $\mathbb{E}^3$ and is generated by two half-turns in $\mathbb{E}^3$.)

**Just five regular nets:** net $\text{srs}$ of Strontium Silicide ($\text{SrSi}_2$), net $\text{nbo}$ of Niobium Monoxide ($\text{NbO}$), diamond net $\text{dia}$, primitive cubic net $\text{pcu}$, and body-centered cubic net $\text{bcu}$. Coordination figures are triangles, squares, tetrahedra, octahedra, or cubes, resp.
**Quasi-regular net** $\mathcal{N}$

- $\mathcal{N}$ is uninodal,

- coordination figures are quasi-regular polyhedra (alternating semi-regular),

**Just one quasi-regular net:** face-centered cubic net $\text{fcu}$, with coordination figure a cuboctahedron.

(O’Keeffe, Hyde, Proserpio, Delgado-Friedrichs, ....)

**Edge graphs of all regular polygonal complexes which are not polyhedra are regular or quasiregular nets.** (The regular net $\text{srs}$ does not occur in this but all others do. But $\text{srs}$ is the net of two regular polyhedra with helical faces.)
..... The End ..... 

Thank you
Abstract The study of highly symmetric structures in Euclidean 3-space has a long and fascinating history tracing back to the early days of geometry. With the passage of time, various notions of polyhedral structures have attracted attention and have brought to light new exciting figures intimately related to finite or infinite groups of isometries. A radically new, skeletal approach to polyhedra was pioneered by Grunbaum in the 1970’s building on Coxeter’s work. A polyhedron is viewed not as a solid but rather as a finite or infinite periodic geometric edge graph in space, equipped with additional polyhedral super-structure imposed by the faces. Since the mid 1970’s there has been a lot of activity in this area. The lecture surveys the present state of the ongoing program to classify discrete polyhedral structures in ordinary space by symmetry, where the degree of
symmetry is defined via distinguished transitivity properties of the geometric symmetry groups. These skeletal figures exhibit fascinating geometric, combinatorial, and algebraic properties and include many new finite polyhedra as well as many new periodic structures with crystallographic symmetry groups.
Uniform Polyhedra $P$ in $\mathbb{E}^3$

- $G(P)$ transitive on vertices of $P$.
- Faces are *regular* polygons (flat, skew, helical, zigzag).

**What is known?**

- **Convex** polyhedra: Archimedean solids
  Skeletal analogues of the Archimedean solids!

- **Finite** uniform polyhedra with *planar* faces:
  - Classical paper by Coxeter, Longuet-Higgins and Miller (1954)

- **Arbitrary** uniform skeletal polyhedra wide open!
  - Finite polyhedra with skew faces not classified.
Tractable class: Wythoffians (“truncations”)
(joint work with A. Williams)

Observations:
- Archimedean solids from Platonic solids via Wythoff’s construction! Wythoff exploits reflection group structure!
- Archimedean solids: uniform Wythoffians of Platonics!
- All 48 regular skeletal polyhedra in $\mathbb{E}^3$ have groups generated by reflections (in points, lines or planes)!
- Exploit reflection group structure! Run Wythoff to produce skeletal Wythoffians of the regular skeletal polyhedra.
- Initial vertex in $\mathbb{E}^3$ (not on a surface)! Less confined than in the traditional case.
- Wythoffians always vertex-transitive! Many examples of skeletal uniform polyhedra!